Which classes of origin graphs are generated by transducers?

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Definition

A transduction is a subset of $\Sigma^* \times \Gamma^*$. 
Definition

A transduction is a subset of $\Sigma^* \times \Gamma^*$.

An origin transduction is a set of origin graphs:

```
 a b c a b b a c b
```

Input edge

```
 a a a b b b b
```

Output edge

```
 a a a b b b b
```

Origin color

```
 a b c a b b a c b
```

Input edge

```
 a a a b b b b
```

Output edge
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a | w | a . b | w | b \]
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

**Example**

$$w \mapsto a|w|_a \cdot b|w|_b$$

![Diagram](image)
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a \cdot w \cdot b \]

\[
\begin{array}{cccccccc}
\top & a & b & c & a & b & b & a & c & b & \bot \\
\end{array}
\]

A

read

write

input tape

output tape

Remark

We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
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Example

\[ w \mapsto a | w | a \cdot b | w | b \]

| ← | a | b | c | a | b | b | a | c | b | ← |

\[ A \]

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- 2-way automata with outputs (2FT);
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Example

\( w \mapsto a|w|_a \cdot b|w|_b \)

![Diagram showing 2-way automata and streaming string transducer with input and output symbols, and a transition example.]

Remark

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Origin semantics of transducers

- 2-way automata with outputs (2FT);
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Example

\[ w \mapsto \overline{a|w|_a \cdot b|w|_b} \]

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Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a \cdot b \]

![Diagram of a transducer with input 'aabbaba' and output 'abaab']
Origin semantics of transducers

- 2-way automata with outputs ($2\text{FT}$);
- streaming string transducer ($\text{SST}$);
- MSO transductions ($\text{MSOT}$).

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$$w \mapsto a|w|_a \cdot b|w|_b$$

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Example

\[ w \mapsto a \lvert \cdot \rvert b \]

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We have an MSO transduction from input word to origin graphs.

\[ \vdash \]
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[
\begin{array}{c}
  \text{read input tape} \\
  \text{write output tape}
\end{array}
\]

\[
| \quad \begin{array}{cccccccc}
  a & b & c & a & b & b & a & c & b & \mid
  \\
  a & a & a & b & b & b & b & \mid
\end{array}
\]

Remark
We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- Streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a|w|_a \cdot b|w|_b \]

\[ \begin{array}{c}
\text{read} \\
\text{input tape} \\
\text{write} \\
\text{output tape}
\end{array} \]

\[ \begin{array}{c}
a \rightarrow b \\
a \rightarrow c \\
a \rightarrow a \\
b \rightarrow a \\
b \rightarrow b \\
b \rightarrow b \\
b \rightarrow b \\
c \rightarrow b
\end{array} \]

Remark

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Origin semantics of transducers

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Example

$$w \mapsto a \cdot b$$

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Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a|w_a \cdot b|w_b \]

\[
\begin{array}{cccccccc}
  a & b & c & a & b & b & a & c & b \\
\end{array}
\]

Register 1: \[ A \]

Register 2:
2-way automata with outputs ($2FT$);
streaming string transducer ($SST$);
MSO transductions ($MSOT$).

Example

$$w \mapsto a^{\|w\|_a} \cdot b^{\|w\|_b}$$
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

**Example**

\[ \omega \mapsto a^{|\omega|_a} \cdot b^{|\omega|_b} \]

<table>
<thead>
<tr>
<th>register 1:</th>
<th>register 2:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>b</td>
</tr>
</tbody>
</table>

Remark: We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs ($2FT$);
- streaming string transducer ($SST$);
- MSO transductions ($MSOT$).

Example

$$w \mapsto a^{w_a} \cdot b^{w_b}$$

Remark

We have an $MSO$ transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs ($2FT$);
- streaming string transducer ($SST$);
- MSO transductions ($MSOT$).

**Example**

$$w \mapsto a^{|w|_a} \cdot b^{|w|_b}$$

**Diagram**

```
+---+---+---+---+---+---+---+---+
| a | b | c | a | b | b | a | c | b |
+---+---+---+---+---+---+---+---+
```

**Register 1:**

```
+---+---+
| a | a |
+---+---+
```

**Register 2:**

```
+---+
| b |
+---+
```

**Remark**

We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs ($2\text{FT}$);
- streaming string transducer ($\text{SST}$);
- MSO transductions ($\text{MSOT}$).

Example

$w \mapsto a^{\lfloor w \rfloor_a} \cdot b^{\lfloor w \rfloor_b}$

Remark

We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs ($2FT$);
- streaming string transducer ($SST$);
- MSO transductions ($MSOT$).

Example

$$w \mapsto a \cdot w_a \cdot b \cdot w_b$$

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>a</th>
<th>b</th>
<th>b</th>
<th>a</th>
<th>c</th>
<th>b</th>
<th>--</th>
</tr>
</thead>
</table>

register 1:

```
A
a a a b b b b
```

register 2:

```
A
b b b b
```

Remark

We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs (2FT);
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Example

\[ w \mapsto a^{w|_a} \cdot b^{w|_b} \]

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We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs ($2FT$);
- streaming string transducer ($SSST$);
- MSO transductions ($MSOT$).

Example

\[ w \mapsto a | w_a \cdot b | w_b \]

\[ \begin{array}{c}
  \text{register 1:} \\
  a \quad a \quad a \\
  \text{register 2:} \\
  b \quad b \quad b \\
\end{array} \]

Remark
We have an MSO transduction from input word to origin graphs.
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\[ w \mapsto a^{|w|_a} \cdot b^{|w|_b} \]

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Example

\[ w \mapsto a \upharpoonright w \upharpoonright_a b \upharpoonright w \upharpoonright_b \]

Remark

We have an MSO transduction from input word to origin graphs.
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Example

$$w \mapsto a|w|_a \cdot b|w|_b$$

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**Example**

\[ w \mapsto \begin{array}{c}
  a | w_a \cdot b | w_b \\
\end{array} \]

**Diagram:****

```
  register 1: [a] [a] [a] [b] [b] [b] [b] 
  register 2: [ ]

  output

  A
```
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

**Example**

\[ w \mapsto a^{\mid w \mid_a} \cdot b^{\mid w \mid_b} \]

![Diagram showing the transduction process with input and output sequences.](image_url)
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a_w^a \cdot b_w^b \]
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \mapsto a|w|_a . b|w|_b \]

![Diagram showing the transformation of a string w through transducer]
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

**Example**

\[ w \mapsto a|w|_a \cdot b|w|_b \]
Origin semantics of transducers

- 2-way automata with outputs (2FT);
- streaming string transducer (SST);
- MSO transductions (MSOT).

Example

\[ w \rightarrow a^{\mid w \mid_a} \cdot b^{\mid w \mid_b} \]

\[ a \rightarrow b \rightarrow c \rightarrow a \rightarrow b \rightarrow b \rightarrow a \rightarrow c \rightarrow b \]

Copy 1

Copy 2
Origin semantics of transducers

- 2-way automata with outputs ($2\text{FT}$);
- streaming string transducer ($\text{SST}$);
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**Example**

\[ w \mapsto a^{|w|_a} \cdot b^{|w|_b} \]

Remark

We have an MSO transduction from input word to origin graphs.
Origin semantics of transducers

- 2-way automata with outputs ($2FT$);
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Example

$$w \mapsto a \| w \|_a \cdot b \| w \|_b$$

Remark

We have an MSO transduction from input word to origin graphs.
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- 2-way automata with outputs ($2\text{FT}$);
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- 2-way automata with outputs ($2\text{FT}$);
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**Example**

$$w \mapsto a^{w_a} \cdot b^{w_b}$$

**Remark**

*We have an MSO transduction from input word to origin graphs.*
MSO logic is undecidable on unrestricted origin transductions.
Questions

MSO logic is undecidable on MSO-definable origin transductions.
MSO logic is undecidable on **MSO-definable** origin transductions.

An **MSO** sentence over the *origin vocabulary*:

- predicates for *input*, *output* and *origin* edges;
- predicates for \((\Sigma \cup \Gamma)\)-labellings.
MSO logic is undecidable on MSO-definable origin transductions.

An MSO sentence over the origin vocabulary:
- predicates for input, output and origin edges;
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Questions

- *Is MSO decidable on origin semantics of transducers?*
MSO logic is undecidable on **MSO-definable** origin transductions.

An MSO sentence over the *origin vocabulary*:
- predicates for input, output and origin edges;
- predicates for \((\Sigma \cup \Gamma)\)-labellings.

**Questions**

- *Is MSO decidable on origin semantics of transducers?*
- *Which origin transductions are realised by transducer?*
Theorem

The following is **decidable**:

**Input**
- a transducer $\mathcal{A}$
- an MSO formula $\phi$ over the corresponding origin vocabulary

**Question**
- Is $\phi$ true in some origin graph in the origin semantics of $\mathcal{A}$?
The following is **decidable**:

**Input**
- a transducer $A$
- an MSO formula $\phi$ over the corresponding origin vocabulary

**Question**
- Is $\phi$ true in some origin graph in the origin semantics of $A$?

**Example**

“The output may be split in two parts such that the origin mapping is order-preserving on each part.”
MSO satisfiability on origin semantics

Theorem

The following is **decidable**:

**Input**
- a transducer $A$
- an MSO formula $\phi$ over the corresponding origin vocabulary

**Question**
- Is $\phi$ true in some origin graph in the origin semantics of $A$?

**Proof.**

- from a string-to-string MSO-transduction, we can obtain a string-to-origin graph MSO-transduction
Theorem

The following is **decidable**:

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Consider $L = \left\{ w \in \Sigma^* \mid \phi \text{ is true in the origin graph produced by } A \text{ on } w \right\}$
Theorem

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Consider $L = \left\{ w \in \Sigma^* \mid \phi \text{ is true in the origin graph produced by } A \text{ on } w \right\}$

by Backward Translation Theorem [Courcelle&Engelfriet 2012], $L$ is regular.
Theorem

The following is decidable:

Input

- a transducer $A$
- an MSO formula $\phi$ over the corresponding origin vocabulary

Question

- Is $\phi$ true in some origin graph in the origin semantics of $A$?

linear time when $A$ is fixed

Proof.

- from a string-to-string MSO-transduction we can obtain a string-to-origin graph MSO-transduction

Consider $L = \left\{ w \in \Sigma^* \mid \phi \text{ is true in the origin graph produced by } A \text{ on } w \right\}$

by Backward Translation Theorem [Courcelle&Engelfriet 2012], $L$ is regular.
Theorem
An origin transduction is the origin semantics of a functional $ss$ if and only if it is $mso$-definable over origin vocabulary:
- functional: for each input word, there is at most one origin graph;
- bounded origin: each input position is the origin of at most $m$ output positions;
- bounded crossing: next slide.
Theorem

An *origin transduction* is the *origin semantics* of a functional SST if and only if it is
Theorem

An origin transduction is the origin semantics of a functional SST if and only if it is

- MSO-definable over origin vocabulary;
Theorem

An origin transduction is the origin semantics of a functional SST if and only if it is

- MSO-definable over origin vocabulary;
- functional:
Which origin transduction is an origin regular transduction?

Theorem

An origin transduction is the origin semantics of a functional SST if and only if it is

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An origin transduction is the origin semantics of a functional SST if and only if it is

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An origin transduction is the origin semantics of a functional SST if and only if it is

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  for each input word, there is at most one origin graph;
- bounded origin:
  each input position is the origin of at most m output positions;
- bounded crossing:

NEXT SLIDE.
**Definition (crossing)**

- crossing of an input position
  - number of maximal factors of the output that originate in the input prefix ending in the position

![Diagram of crossing example](image-url)
Definition (crossing)

- crossing of an input position
  number of maximal factors of the output
  that originate in the input prefix ending in the position

```latex
\begin{align*}
\text{a} & \rightarrow \text{b} & \rightarrow \text{c} \quad \text{a} \rightarrow \text{b} & \rightarrow \text{b} \\
\text{a} & \rightarrow \text{a} & \rightarrow \text{a} \quad \text{b} & \rightarrow \text{b} & \rightarrow \text{b} & \rightarrow \text{b}
\end{align*}
```
Definition (crossing)

crossing of an input position

number of maximal factors of the output

that originate in the input prefix ending in the position

```
a b c a b b a c b
```

```
left 1 left 2 right 1 right 2
```

crossing: 2

\[
\frac{7}{11}
\]
**Definition (crossing)**

- crossing of an **input position**
- number of maximal factors of the **output**
  that originate in the **input prefix** ending in the position

---

Diagram:

```
  a b c a b b a c b
left1 left2 right1 right2
```

crossing: 2
Definition (crossing)

- crossing of an input position
  number of maximal factors of the output
  that originate in the input prefix ending in the position
- crossing of an origin graph: \( \text{max} \) of the crossings

```
 crossing: 2
```

```
left_1

right_1

left_2

right_2
```
Theorem (with nondeterminism?)

A set of origin graphs is

- the origin semantics of a \textit{k-register unambiguous} SST
  if and only if it is

  MSO-definable, functional and \textit{k-bounded crossing}
A set of origin graphs is
- the origin semantics of a \textit{k}-register \textbf{unambiguous} SST
  if and only if it is
  MSO-definable, functional \textbf{and} \textit{k}-bounded crossing

Remark:
\[
\{ \text{MSO-definable, functional, bounded crossing} \} \implies \text{bounded origin}
\]
Theorem (with nondeterminism?)

A set of origin graphs is

- the origin semantics of a \( k \)-register unambiguous SST if and only if it is
  \( \text{MSO-definable, functional and } k \)-bounded crossing

- the origin semantics of a \( k \)-register nondeterministic SST if and only if it is
  \( \text{MSO-definable, bounded origin and } k \)-bounded crossing

Remark:

\[
\begin{align*}
\text{MSO-definable} & \quad \text{functional} \\
\text{bounded crossing} & \quad \implies \quad \text{bounded origin}
\end{align*}
\]
Theorem (with nondeterminism?)

A set of origin graphs is

- the origin semantics of a k-register \textbf{unambiguous} SST if and only if it is MSO-definable, functional and k-bounded crossing

- the origin semantics of a k-register \textbf{nondeterministic} SST if and only if it is MSO-definable, bounded origin and k-bounded crossing

An \textbf{MSO-definable set of origin graphs is}

- the origin semantics of a k-register \textbf{NSST with } ε-transitions if and only if it is \textbf{k-bounded crossing}

Remark: \space MSO-definable \hspace{0.5cm} \begin{array}{l} \hspace{1.5cm} \text{functional} \hspace{2cm} \text{bounded crossing} \end{array} \hspace{0.5cm} \rightarrow \hspace{0.5cm} \text{bounded origin}
Sketch of the proof \(\implies\)

- unambiguous \(\implies\) functional
- NSST \(\implies\) bounded origin
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
- NSST $\implies$ nondeterministic MSO-transduction $\implies$ MSO-definable
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
- NSST $\implies$ nondeterministic MSO-transduction

$\implies$ MSO-definable
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
- NSST $\implies$ nondeterministic MSO-transduction $\implies$ MSO-definable

\[ \rho \]
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
- NSST $\implies$ nondeterministic MSO-transduction

$\implies$ MSO-definable
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
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$\implies$ MSO-definable
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
- NSST $\implies$ nondeterministic MSO-transduction

$\implies$ MSO-definable
Sketch of the proof \( \Longrightarrow \)

- unambiguous \( \Longrightarrow \) functional
- NSST \( \Longrightarrow \) bounded origin
- NSST \( \Longrightarrow \) nondeterministic MSO-transduction
  \( \Longrightarrow \) MSO-definable

False when \( \varepsilon \)-transitions are allowed.
Sketch of the proof

- unambiguous $\Rightarrow$ functional
- NSST $\Rightarrow$ bounded origin
- NSST $\Rightarrow$ nondeterministic MSO-transduction $\Rightarrow$ MSO-definable
- $k$-register $\Rightarrow$ $k$-bounded crossing
Sketch of the proof

- unambiguous $\implies$ functional
- NSST $\implies$ bounded origin
- NSST $\implies$ nondeterministic MSO-transduction $\implies$ MSO-definable
- $k$-register $\implies$ $k$-bounded crossing

Diagram:

- Left 1
- Right 1
- Left 2
- Right 2
Start with an \textit{MSO-definable} set of origin graphs $G$ with crossing bounded by $k$. 

Definition ($k$-block origin graphs ($k$-BLOGs))

An origin graph with output split in $k$ identified blocks.
Start with an **MSO-definable** set of origin graphs $G$ with crossing bounded by $k$.

**Definition (k-block origin graphs (k-BLOGs))**

An origin graph with **output** split in $k$ identified blocks.
Sketch of the proof \(\iff\)

- Start with an \textbf{MSO-definable} set of origin graphs \(G\) with crossing bounded by \(k\)

**Definition \((k\text{-block origin graphs \((k\text{-BLOGs})\))**

An origin graph with output split in \(k\) identified blocks.

![Graph Diagram]

\[\text{Definition \((k\text{-block origin graphs \((k\text{-BLOGs})\))}}\]
Start with an MSO-definable set of origin graphs $G$ with crossing bounded by $k$.

Definition ($k$-block origin graphs ($k$-BLOGs))

An origin graph with output split in $k$ identified blocks.
Start with an **MSO-definable** set of origin graphs $G$ with crossing bounded by $k$

we define a finite set of (partial) operations $\Omega_k$ on $k$-BLOGs

**Definition (k-block origin graphs (k-BLOGs))**

An origin graph with output split in $k$ identified blocks.

![Diagram of k-block origin graphs](image)
Sketch of the proof

- Start with an **MSO-definable** set of origin graphs $G$
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- we define a finite set of (partial) operations $\Omega_k$ on $k$-BLOGs
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- from an automaton recognising $L$,
  we build a NSST with $\varepsilon$-transitions realising $G$
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